PROBLEM SET 02

MARC PICKETT I

(1) What is the negation of the statement "All women are strong, all men are handsome, and all children are above average."?

Answer: "Some women aren't strong, some men aren't handsome, *or* some children aren't above average."

(2) Prove (or disprove) the following statements. *State which proof technique you used.*

(a) "The sum of 2 consecutive integers is odd."

Answer: Here we can do a *direct proof*: Any 2 consecutive integers can be expressed as n (where n is an integer) and $n+1$. The sum is then $2n+1$, which is odd, since $2n$ is even.

(b) "The sum of 5 consecutive integers is odd."

Answer: We can *disprove* this by *counterexample*: namely, $2+3+4+5+6=20$, which is even.

(c) "The sum of 5 consecutive integers is evenly divisible by 5."

Answer: We can use a *direct proof* again as in part a: The sum of any 5 consecutive numbers can be expressed as $n+(n+1)+(n+2)+(n+3)+(n+4)=5n+10=5(n+2)$ which is evenly divisible by 5.

(3) What can we say about sets A and B if:

(a) $A \cup B = A$

Answer: $B \subseteq A$

(b) $A - B = \emptyset$

Answer: $A \subseteq B$

(c) $|A \cup B| = |A| + |B| - |A \cap B|$

Answer: We can't say anything about A and B because this is true for all sets.

(4) Translate the following Set statements into Logic. For example, the proposition $A \cup B \subseteq C$ would be $\forall x\,((x \in A \vee x \in B) \implies x \in C)$.

(a) $(A \cap B) \subseteq E$

Answer: $\forall x\,((x \in A \wedge x \in B) \implies x \in E)$

(For these problems, note the relation between $\cup$ and $\vee$, $\cap$ and $\wedge$, $\subseteq$ and $\implies$, $-$ and $\sim$, and $=$ and $\iff$.)

(b) $(A \cap B) = \emptyset$

Answer: $\forall x \sim (x \in A \wedge x \in B)$

(c) $(A \cap B) \subseteq (C - D)$

Answer: $\forall x\,((x \in A \wedge x \in B) \implies (x \in C \wedge \sim x \in D))$

(d) $(A \cap B) \cup (C - D) = E$

Answer: $\forall x\,(((x \in A \wedge x \in B) \vee (x \in C \wedge \sim x \in D)) \iff (x \in E))$

(e) $A \cup B \subset C$ (Note the use of $\subset$ instead of $\subseteq$)

Answer: $\forall x\,((x \in A \vee x \in B) \implies x \in C) \wedge \exists y\,(\sim (y \in A \vee y \in B) \wedge y \in C)$

(5) BONUS: Prove or disprove that the product of 2 of the following numbers is non-negative: (The proof must be less than a page to receive credit.)

- $2^{2342} - 8^{780} + 3^{721}$
- $\sqrt{2}^{\sqrt{2}^{3138}} - \sqrt{3}^{\sqrt{3}^{1108}}$
- $999^{888} - 888^{999} + 777^{1020}$

Answer: We can use a *nonconstructive proof by cases* where we don't know which 2 of the numbers we can use to multiply to make a non-negative number. We note that each of these numbers is either positive, or non-positive. Also note that the product of 2 non-positive numbers is non-positive. So, there are 4 cases:

(a) All 3 numbers are positive: Then the product of any 2 numbers is non-negative.

(b) All 3 numbers are non-positive: In which case the product of any 2 numbers is non-negative.

(c) 2 numbers are non-positive and 1 is positive: In which case the product of the 2 non-positive numbers is non-negative.

(d) 2 numbers are positive and 1 is non-positive: In which case the product of the 2 positive number is non-negative.

In every case, we see that the product of 2 of the numbers is non-negative, which is what we set out to prove.