PROBLEM SET 04

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(1) N-year cicadas breed once every N years. How often will the breeding year for 17-year cicadas coincide with that of 13-year cicadas?

Answer: This is simply the least common multiple of 17 and 13. Since both 17 and 13 are prime, this is $17 \cdot 13$ or **every 221 years**.

(2) What is 14600926_{10} in hexadecimal (base 16)?

Answer: To convert this to hexadecimal, we note that

$$16^0 = 1 \quad (1)$$
$$16^1 = 16 \quad (2)$$
$$16^2 = 256 \quad (3)$$
$$16^3 = 4,096 \quad (4)$$
$$16^4 = 65,536 \quad (5)$$
$$16^5 = 1,048,576 \quad (6)$$
$$16^6 = 16,777,216 \quad (7)$$

Since $16^6 = 16,777,216 > 14,600,926$ we know that our answer will be $A_5A_4A_3A_2A_1A_0$, where

$$14,600,926 = A_5 16^5 + A_4 16^4 + A_3 16^3 + A_2 16^2 + A_1 16^1 + A_0$$

To find out what A_5 must be (i.e., to see how many times 16^5 goes into 14,600,926), we divide 14,600,926 by $16^5 = 1,048,576$, and we get 13, remainder 969,438. Now we know that

$$14,600,926 = 13 \cdot 16^5 + A_4 16^4 + A_3 16^3 + A_2 16^2 + A_1 16^1 + A_0$$

We can subtract $13 \cdot 16^5$ from both sides to get a new equation:

$$969,438 = A_4 16^4 + A_3 16^3 + A_2 16^2 + A_1 16^1 + A_0$$

Now we divide 969,438 by 16^4 to get 14 remainder 51,934. So we have the new equation.

$$14,600,926 = 13 \cdot 16^5 + 14 \cdot 16^4 + A_3 16^3 + A_2 16^2 + A_1 16^1 + A_0$$

or

$$51,934 = A_3 16^3 + A_2 16^2 + A_1 16^1 + A_0$$

We repeat this process until we get

$$14,600,926 = 13 \cdot 16^5 + 14 \cdot 16^4 + 12 \cdot 16^3 + 10 \cdot 16^2 + 13 \cdot 16^1 + 14$$

We now substitute the hexadecimal characters $A = 10, B = 11$, etc. to get our answer $14,600,926_{10} = DECADE_{16}$.

(3) (a) What is the the prime factorization of 18!? (Note, that's 18 factorial, not just 18.)

Answer: We can solve this by factorizing all the terms of 18!, then adding the exponents of the prime factors:

$$18! = 18 \cdot 17 \cdot 16 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot \tag{8}$$

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \tag{9}$$

$$= \left(3^2 \cdot 2^1\right) \cdot 17^1 \cdot 2^4 \cdot \left(3^1 \cdot 5^1\right) \cdot \left(2^1 \cdot 7^1\right) \cdot 13^1 \cdot \left(2^2 \cdot 3^1\right) \cdot 11^1 \cdot \tag{10}$$

$$\left(2^1 \cdot 5^1\right) \cdot 3^2 \cdot 2^3 \cdot 7^1 \cdot \left(2^1 \cdot 3^1\right) \cdot 5^1 \cdot 2^2 \cdot 3^1 \cdot 2^1 \tag{11}$$

$$= 17^1 \cdot 13^1 \cdot 11^1 \cdot 7^1 \cdot 7^1 \cdot 5^1 \cdot 5^1 \cdot 5^1 \cdot 3^2 \cdot 3^1 \cdot 3^2 \cdot \tag{12}$$

$$3^1 \cdot 3^1 \cdot 3^1 \cdot 2^1 \cdot 2^4 \cdot 2^1 \cdot 2^2 \cdot 2^1 \cdot 2^3 \cdot 2^1 \cdot 2^2 \cdot 2^1 \tag{13}$$

$$= 17^1 \cdot 13^1 \cdot 11^1 \cdot 7^2 \cdot 5^3 \cdot 3^8 \cdot 2^{16} \tag{14}$$

So our answer is that $18! = 17^1 \cdot 13^1 \cdot 11^1 \cdot 7^2 \cdot 5^3 \cdot 3^8 \cdot 2^{16}$.

(b) What is the smallest positive integer that's evenly divisible by all integers from 2 to 18?

Answer: This number will be the least common multiple of all the numbers from 2 to 18. To find this, we note that the primes up to 18 are 17, 13, 11, 7, 5, 3, and 2. So our answer will be of the form $17^a \cdot 13^b \cdot 11^c \cdot 7^d \cdot 5^e \cdot 3^f \cdot 2^g$, where $a, b, c, \cdots$ are the highest power of that respective prime in the factorization of the numbers from 2 to 18.

For example, the highest power of 2 in the factorization of any number between 2 and 18 is 4 (for 16, which is 2^4), so we know that g will be 4. Likewise, the highest power for 3 is 2 (so $f = 2$), and the highest power for each of the remaining primes is 1. So our number is

$$17^1 \cdot 13^1 \cdot 11^1 \cdot 7^1 \cdot 5^1 \cdot 3^2 \cdot 2^4 = 12,252,240$$

(4) (a) Write a *closed form* formula (i.e. one with no "$\cdots$" or "$\sum$") for $7 + 14 + 21 + 28 + 35 + \cdots + 7n$.

Answer: The way I solved this is by noting that $7+14+21+28+35+\cdots+7n = 7\,(1 + 2 + \cdots + n)$. Now, all I had to do was find an equation for $1+2+\cdots+n$, then multiply it by 7. To do this, I noted that I can pair up the 1st term and the last term to make $n + 1$, then I can pair up the 2nd term with the 2nd to last term to make another $n + 1$, etc.. If the number of terms is even, then there are $\frac{n}{2}$ such pairings, so I can conclude that $1 + 2 + \cdots + n = \frac{n}{2}\,(n + 1)$. In the case that n is odd, there will be $\frac{n-1}{2}$ such terms, and then I'll need to add in the middle term (which will be $\frac{n+1}{2}$). In this case, the sum will be $\frac{n-1}{2}\,(n + 1) + \frac{n+1}{2} = \frac{(n-1)(n+1)+n+1}{2} = \frac{n^2-1+n+1}{2} = \frac{n^2+n}{2} = \frac{n}{2}\,(n + 1)$, which is the same as the case when n is even. Therefore, I conclude that

$$7 + 14 + 21 + 28 + 35 + \cdots + 7n = 7\frac{n}{2}\,(n + 1)$$

(b) Prove that your answer is valid.

Answer: To double check, I'll use induction to prove my result:

The base case is true: $7\frac{1}{2}\,(1 + 1) = 7$

For the inductive step, we assume that $7+14+21+28+35+\cdots+7n = 7\frac{n}{2}(n+1)$, then, substituting $(n+1)$ for n in $7\frac{n}{2}(n+1)$, we get:

$$\tag{15}$$

$$= 7\frac{(n+1)}{2}((n+1)+1) \tag{16}$$

$$= 7\frac{(n+1)(n+2)}{2} \tag{17}$$

$$= 7\frac{n^2+3n+2}{2} \tag{18}$$

$$= 7\frac{n^2+n+2n+2}{2} \tag{19}$$

$$= \frac{7n^2+7n+14n+14}{2} \tag{20}$$

$$= \frac{7n(n+1)+14(n+1)}{2} \tag{21}$$

$$= 7\frac{n}{2}(n+1)+\frac{2\cdot 7(n+1)}{2} \tag{22}$$

$$= 7\frac{n}{2}(n+1)+7(n+1) \tag{23}$$

$$= (7+14+21+28+35+\cdots+7n)+7(n+1) \tag{24}$$

and we've demonstrated our claim.

(5) (a) Prove or disprove that $4^k - 1$ is evenly divisible by 3 (for all k where k is a positive integer).

Answer: We can prove this using induction:

The base case is true $4^1 - 1 = 3$, which is divisible by 3.

For the inductive step, we assume that $4^k - 1$ is divisible by 3. This means that there's some integer m such that $3m = 4^k - 1$. Now what about $4^{k+1} - 1$? This is

$$4^{k+1} - 1 = 4\cdot 4^k - 1 \tag{25}$$

$$= 3\cdot 4^k + 4^k - 1 \tag{26}$$

$$= 3\cdot 4^k + 3m \tag{27}$$

$$= 3\left(4^k + m\right) \tag{28}$$

which is divisible by 3 since k is an integer. *Quod erat demonstrandum.*

(b) BONUS: Prove or disprove that $N^k - 1$ is evenly divisible by $N - 1$ (for all k and $N - 1$ where k and $N - 1$ are positive integers).

Answer: We can prove this simply by copying the last proof and substituting N for 4 (and making a few other minor modifications), like this:

The base case is true: $N^1 - 1 = N - 1$, which is divisible by $N - 1$.

For the inductive step, we assume that $N^k - 1$ is divisible by $N - 1$. This means that there's some integer m such that $(N - 1)\,m = N^k - 1$. Now what about $N^{k+1} - 1$? This is

$$N^{k+1} - 1 = N \cdot N^k - 1 \tag{29}$$

$$= (N - 1) \cdot N^k + N^k - 1 \tag{30}$$

$$= (N - 1) \cdot N^k + (N - 1)\,m \tag{31}$$

$$= (N - 1)\left(N^k + m\right) \tag{32}$$

which is divisible by $N - 1$ since k is an integer. *Quod erat demonstrandum.*

(Note that we could have used Universal Generalization, that strange principle I touched on in lecture 3 or 4. We generalized the "$4^k - 1$ is divisible by $4 - 1$" theorem by noticing that there were no properties (aside from being an integer) particular to 4 in our proof of that theorem, so we reached the conclusion for all integers $N > 1$.)