PROBLEM SET 06

MARC PICKETT I

(1) How many ways are there to get either a pair of Kings (exactly 2 kings) or a 3-of-a-kind of Aces (or both the King pair and the Ace trio) with a standard poker hand? (For this exercise, a full house with a pair of Kings counts as a pair of Kings, but 3 (or 4) Kings doesn't count as a pair of kings, and 4 Aces doesn't count as a 3-of-a-kind.)

Answer: There are $\binom{4}{2}\binom{(52-4)}{3} = 103,776$ ways to get a pair of Kings, $\binom{4}{3}\binom{(52-4)}{2} = 4,512$ ways to get a 3-of-a-kind of Aces, and there are $\binom{4}{3}\binom{4}{2} = 24$ ways to get a full house with 3 Aces and a pair of Kings. So the total is $103,776 + 4,512 - 24 = 108,264$.

(2) In the expansion for $(A+B)^{114}$, what constant is in front of the term $A^{62}B^{52}$?

Answer: By the Binomial Theorem, this will be $\binom{114}{62}$ which is $1,002,067,957,159,418,145,339,445,454,873,136$ or **just over one decillion**.

(3) Every pack of Starburst© candy has 3 Cherry, 3 Strawberry, 3 Orange and 3 Lemon flavored Starbursts. In our class, 4 people like Cherry the best, 13 people prefer Strawberry, 8 prefer Orange, and 2 prefer Lemon. What is the minimum number of packs of Starbursts I would need to get so that I could divide the Cherry Starbursts evenly among those who prefer Cherry, the Lemon Starbursts evenly among those who prefer Lemon, etc.?

Answer: I'd need to buy enough packs so that the number of Strawberry Starbursts is divisible by 13, the number Orange Starbursts is divisible by 8, etc.. The number of Starbursts of any flavor needs to be divisible by 3 (since they come 3 to a pack), and by 13, 8, 4, and 2. Thus, the number of Starbursts of any flavor will be $LCM(3, 13, 8, 4, 2) = 13^1 3^1 2^3 = 312$. So, I'd need to buy $\frac{312}{3} = 104$ packs.

(4) If I roll 5 standard 6 sided dice, what are the odds that their sum will be 7 or fewer?

Answer: For 5 dice to sum up to 7 or fewer, I can have the following possibilities:

One of the dice comes up 3, and the other 4 dice come up 1: There are $\binom{5}{1} = 5$ ways for this to happen.

Two of the dice come up 2, and the other 3 dice are 1: There are $\binom{5}{2} = 10$ ways for this to occur.

One of the dice comes up 2, and the other 4 dice are 1: There are 5 ways for this.

All the dice are 1: There's only 1 way for this to happen.

So, in total, there are only $5 + 10 + 5 + 1 = 21$ ways to get 7 or less. This is out of $6^5 = 7,776$, so the odds are $\frac{21}{7,776} \approx .0027$.

(5) The incidence of Huntington's disease (an unfortunate genetic disorder) is about 1 in 15,000. There are usually no visible signs for this disease until a person is in their late 30s or early 40s. There is a test for this disease which has a 1.8% false positive rate (i.e., $P\left(TestPositive\left(x\right) | \sim Huntington\left(x\right)\right) = 0.018$ where $TestPositive\left(x\right)$ means that person x gets a positive for the Huntington's test, and $Huntington\left(x\right)$ means that a person actually has Huntington's disease). If a person has Huntington's, the test result will be positive virtually 100% of the time (i.e. $P\left(TestPositive\left(x\right) | Huntington\left(x\right)\right) = 1.0$). Doug was adopted (and therefore we know nothing about occurrence of Huntington's disease in his biological family), and he tests positive for Huntington's.

(a) What is the probability that he actually has this disease?

Answer: (In this answer, I'll abbreviate $TestPositive\left(x\right)$, $Huntington\left(x\right)$, and $Doug$ with $T\left(x\right)$, $H\left(x\right)$, and D, respectively.) The answer is a direct application of Bayes's law:

$$
\begin{aligned}
(1)\quad P\left(H\left(D\right)\right) &= \frac{1}{15,000}\\
(2)\quad P\left(\sim H\left(D\right)\right) &= \frac{14,999}{15,000}\\
(3)\quad P\left(T\left(D\right)|H\left(D\right)\right) &= 1\\
(4)\quad P\left(T\left(D\right)|\sim H\left(D\right)\right) &= 0.018\\
(5)\quad P\left(T\left(D\right)\right) &= P\left(T|H\left(D\right)\right)P\left(H\left(D\right)\right)+P\left(T|\sim H\left(D\right)\right)P\left(\sim H\left(D\right)\right)\\
(6)\quad &= \frac{1}{15,000}+0.018\cdot\frac{14,999}{15,000}\\
(7)\quad P\left(H\left(D\right)|T\left(D\right)\right) &= \frac{P\left(T\left(D\right)|H\left(D\right)\right)P\left(H\left(D\right)\right)}{P\left(T\left(D\right)\right)}\\
(8)\quad &= \frac{\frac{1}{15{,}000}}{\frac{1}{15{,}000}+0.018\cdot\frac{14{,}999}{15{,}000}}\\
(9)\quad &= \frac{1}{1+0.018\cdot 14,999}\\
(10)\quad &\approx .0037
\end{aligned}
$$

So, Doug has about a 1 in 271 chance of having Huntington's, which are less fortunate odds than those for the general populace, but it's still unlikely that Doug has Huntington's disease.

(b) BONUS: The gene that is responsible for Huntington's disease is dominant. This means that the probability of someone inheriting the disease is 50% if exactly one of their parents has the disease (and 100% in the extremely rare case where both of their parents have the disease). Suppose we find out that Doug's biological father is a carrier of this gene, but his biological mother isn't. Given that Doug's test result is positive, what is the probability that he has Huntington's disease?

Answer: Again, we do a direct application of Bayes's Law, but this time we use $P\left(Huntington\left(Doug\right)\right)=\frac{1}{2}$

$$P(H(D)) = \frac{1}{2} \tag{11}$$

$$P(\sim H(D)) = \frac{1}{2} \tag{12}$$

$$P(T(D)|H(D)) = 1 \tag{13}$$

$$P(T(D)|\sim H(D)) = 0.018 \tag{14}$$

$$P(T(D)) = P(T|H(D))\,P(H(D)) + P(T|\sim H(D))\,P(\sim H(D)) \tag{15}$$

$$= \frac{1}{2} + 0.018 \cdot \frac{1}{2} \tag{16}$$

$$P(H(D)|T(D)) = \frac{P(T(D)|H(D))\,P(H(D))}{P(T(D))} \tag{17}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2} + 0.018 \cdot \frac{1}{2}} \tag{18}$$

$$= \frac{1}{1 + 0.018} \tag{19}$$

$$\approx .9823 \tag{20}$$

These odds are much more grim for Doug. Now there's only a 1 in 56 chance that the test results are a false positive.