

PROBLEM SET 07

MARC PICKETT I

- (1) In Marc's casino, I have a game called Mediocre Poker. In this game, I shuffle a deck and (fairly) deal you 5 cards. I take the highest value of your hand and I the pay out *one* of the following: **Pair or less** (Nothing), **2 Pair** \$2, **3 of a kind** \$5, **Straight** \$25, **Flush** \$50, **Full House** \$100, **4 of a kind** \$500, **Straight flush** \$10,000, **Royal Flush** \$100,000. (Remember from the class handout that there are 123,552 ways to get a 2 Pair, 54,912 possible 3 of a kinds, 9,180 Straights, 5,112 Flushes, 3,744 Full Houses, 624 possible 4 of a kinds, 32 Straight flushes, and 4 Royal Flushes.)

2598960 2,401,800

123552 . + 54912 . + 9180 . + 5112 . + 3744 . + 624 . + 32 . + 4

- (a) What is the expected value of playing one game of Mediocre Poker?

Answer: If we note that there are $\binom{52}{5} = 2,598,960$ possible poker hands. We can get the expected value by a direct application of our formula:

$$\begin{aligned}
 (1) \quad E[X] &= \$2P(\mathbf{2\ Pair}) + \$5P(\mathbf{3\ of\ a\ kind}) + \$25P(\mathbf{Straight}) \\
 (2) &\quad + \$50P(\mathbf{Flush}) + \$100P(\mathbf{Full\ House}) + \$500P(\mathbf{4\ of\ a\ kind}) \\
 (3) &\quad + \$10,000P(\mathbf{Straight\ flush}) + \$100,000P(\mathbf{Royal\ Flush}) \\
 (4) &= \$2\frac{123,552}{2,598,960} + \$5\frac{54,912}{2,598,960} + \$25\frac{9,180}{2,598,960} \\
 (5) &\quad + \$50\frac{5,112}{2,598,960} + \$100\frac{3,744}{2,598,960} + \$500\frac{624}{2,598,960} \\
 (6) &\quad + \$10,000\frac{32}{2,598,960} + \$100,000\frac{4}{2,598,960} \\
 (7) &\approx \$0.92851
 \end{aligned}$$

- (b) What is the variance of playing one game of Mediocre Poker?

Answer: Another direct application of our definition of variance:

$$\begin{aligned}
 (8) \quad & E[(X - E[X])^2] \\
 (9) \quad &= (\$0 - E[X])^2 P(\mathbf{Pair\ or\ Less\ (aka\ None\ of\ the\ below)}) \\
 (10) \quad & (\$2 - E[X])^2 P(\mathbf{2\ Pair}) + (\$5 - E[X])^2 P(\mathbf{3\ of\ a\ kind}) \\
 (11) \quad & + (\$25 - E[X])^2 P(\mathbf{Straight}) + (\$50 - E[X])^2 P(\mathbf{Flush}) \\
 (12) \quad & + (\$100 - E[X])^2 P(\mathbf{Full\ House}) + (\$500 - E[X])^2 P(\mathbf{4\ of\ a\ kind}) \\
 (13) \quad & + (\$10,000 - E[X])^2 P(\mathbf{Straight\ flush}) \\
 (14) \quad & + (\$100,000 - E[X])^2 P(\mathbf{Royal\ Flush}) \\
 (15) \quad &= (\$0 - E[X])^2 \frac{2,401,800}{2,598,960} \\
 (16) \quad & (\$2 - E[X])^2 \frac{123,552}{2,598,960} + (\$5 - E[X])^2 \frac{54,912}{2,598,960} \\
 (17) \quad & + (\$25 - E[X])^2 \frac{9,180}{2,598,960} + (\$50 - E[X])^2 \frac{5,112}{2,598,960} \\
 (18) \quad & + (\$100 - E[X])^2 \frac{3,744}{2,598,960} + (\$500 - E[X])^2 \frac{624}{2,598,960} \\
 (19) \quad & + (\$10,000 - E[X])^2 \frac{32}{2,598,960} \\
 (20) \quad & + (\$100,000 - E[X])^2 \frac{4}{2,598,960} \\
 (21) \quad & \approx \$^2 16,703.4443
 \end{aligned}$$

Note that the variance would be in units of square dollars. (I don't know what that means except in a mathematical sense.) The standard deviation would be $\sqrt{V[X]} \approx \$129.24$.

(2) In Marc's Casino, I've decided to automate Mediocre Poker using a coin operated machine. My machine accepts nickels, dimes, and quarters.

(a) Set up a recurrence relation for the number of different ways one can pay $5n$ cents to the machine (where the order in which the coins are inserted matters).

Answer: There is only 1 way we can insert 0 cents, and only 1 way to insert 5 cents (a nickel). There are 2 ways to insert 10 cents (2 nickels or a dime), 3 ways

to insert 15 cents (3 nickels, a nickel followed by a dime, and dime followed by a nickel), and 5 ways to insert 20 cents. (This can be figured by listing out all the possibilities, or by setting up a recurrence relation with just nickels and dimes.) So the base cases of our recurrence relation are: $a_0 = 1$ $a_5 = 1$ $a_{10} = 2$ $a_{15} = 3$ $a_{20} = 5$.

If $5n$ is 25 or above (say 30), then the last coin added can be either a quarter (to 5 cents), a dime (to 20 cents), or a nickel (to 25 cents). So we have all the possibilities that there were starting with a quarter less than $5n$ (i.e., a_{5n-25}), plus all the possibilities starting with a dime less than $5n$ (i.e., a_{5n-10}) plus those starting with a nickel less than $5n$.

This means that our recurrence relation would be

$$a_{5n} = a_{5n-25} + a_{5n-10} + a_{5n-5}$$

(Note that a_m is undefined if m isn't divisible by 5.)

- (b) What characteristic equation would we have to solve to obtain a closed form solution for the above recurrence relation?

Answer: If we define $b_n = a_{5n}$, then our recurrence relation in terms of b becomes

$$b_n = b_{n-5} + b_{n-2} + b_{n-1}$$

So we can set up the characteristic equation for b by setting

$$r^n = r^{n-1} + r^{n-2} + r^{n-5}$$

then dividing by r^{n-5} and rearranging to get

$$r^5 - r^4 - r^3 - 1 = 0$$

Alternatively, we can give the characteristic equation for a by noting that $a_n = a_{n-25} + a_{n-10} + a_{n-5}$, so we get

$$t^n = t^{n-5} + t^{n-10} + t^{n-25}$$

then dividing by t^{n-25} and rearranging to get

$$t^{25} - t^{20} - t^{15} - 1 = 0$$

- (c) If it costs \$1 to play Mediocre Poker, how many ways are there to feed the machine?

Answer: Using the above recurrence relation, we just have to compute a_{100} . (I wrote a short program to do this, but it'd take only a few minutes using a calculator): $a_0 = 1$, $a_5 = 1$, $a_{10} = 2$, $a_{15} = 3$, $a_{20} = 5$, $a_{25} = 9$, $a_{30} = 15$, $a_{35} = 26$, $a_{40} = 44$, $a_{45} = 75$, $a_{50} = 128$, $a_{55} = 218$, $a_{60} = 372$, $a_{65} = 634$, $a_{70} = 1,081$, $a_{75} = 1,843$, $a_{80} = 3,142$, $a_{85} = 5,357$, $a_{90} = 9,133$, $a_{95} = 15,571$, and $a_{100} = 26,547$.

So our answer is 26,547 ways.

- (d) How many ways would there be to insert 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 cents, respectively, if my Mediocre Poker machine took only 3-cent-pieces and pennies?

Answer: Following similar reasoning to part a, we can set up the recurrence relation: $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-3}$. So we get that the 0th through 10th terms are: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, and 28 ways to insert 0 through 10 cents.

- (3) (a) List all 16 of the relations on $\{0, 1\}$:

Answer: In no particular order, they are:

- (i) $\{(1, 1)\}$
- (ii) $\{(1, 0) (0, 1) (0, 0)\}$
- (iii) $\{(1, 1) (0, 1) (0, 0)\}$
- (iv) $\{(1, 1) (1, 0) (0, 0)\}$
- (v) $\{(1, 1) (0, 0)\}$
- (vi) $\{(1, 0) (0, 1)\}$
- (vii) $\{(1, 1) (1, 0) (0, 1)\}$
- (viii) $\{(0, 0)\}$

(ix) $\{(1, 1) (1, 0) (0, 1) (0, 0)\}$

(x) $\{\}$

(xi) $\{(1, 1) (1, 0)\}$

(xii) $\{(1, 1) (0, 1)\}$

(xiii) $\{(0, 1) (0, 0)\}$

(xiv) $\{(1, 0) (0, 0)\}$

(xv) $\{(0, 1)\}$

(xvi) $\{(1, 0)\}$

(b) BONUS: Name all of the 16 relations on $\{True, False\}$ (e.g., “and”, “only if”, “nand”):

Answer: Abbreviating T for *True* and F for *false*, the 16 relations are

and: $\{(T, T)\}$

nand: $\{(T, F) (F, T) (F, F)\}$

only if (or implies): $\{(T, T) (F, T) (F, F)\}$

if: $\{(T, T) (T, F) (F, F)\}$

if and only if: $\{(T, T) (F, F)\}$

xor: $\{(T, F) (F, T)\}$

or: $\{(T, T) (T, F) (F, T)\}$

nor: $\{(F, F)\}$

true (or tautology): $\{(T, T) (T, F) (F, T) (F, F)\}$

false (or contradiction): $\{\}$

“the former”: $\{(T, T) (T, F)\}$

“the latter”: $\{(T, T) (F, T)\}$

“not the former”: $\{(F, T) (F, F)\}$

“not the latter”: $\{(T, F) (F, F)\}$

nif (not if): $\{(F, T)\}$

nimp (not implies): $\{(T, F)\}$