

## PROBLEM SET 08

MARC PICKETT I

**NOTE:** This is due Wednesday, November 23rd, at 4PM. You may either turn it in in class on Tuesday, or submit it via email (marc@coral-lab.org). If you email it to me, I must receive it before 4PM Wednesday (at which point I'll be posting the answers online, and leaving to eat turkey). For the email submissions, I prefer pdf, but if you can't figure out how to make pdf, I'll accept Word docs, rtf, LaTeX source, and plain old text. (If there's another format that you plan to use that I don't have listed, let me know.)

- (1) Kompleck City is a square 9 blocks on a side. On the northeast corner is Vlad Urr's Royal Shack, and on the southwest corner is The Great Library of Kompleck City. There are 21 streets in Kompleck City. These are all one-way streets. One of the streets, called "The Zim Bob Way", goes underground from Count Urr's Royal Shack to The Great Library. The remaining 20 streets are arranged in a 10 by 10 grid. 10 of these streets (Ave.  $X_0$  through Ave.  $X_9$ ) go from south to north, and 10 (Ave.  $Y_0$  through Ave.  $Y_9$ ) go from west to east. For every day of his reign, Count Urr goes to the Royal Library (along The Zim Bob Way), then returns along a different path to survey his realm.

(An example path would be where Count Urr takes Zim Bob Way to The Great Library (at the intersection of  $X_0$  and  $Y_0$ ), then Ave.  $X_0$  to Ave.  $Y_1$ , then  $Y_1$  all the way to  $X_3$ , then  $X_3$  to  $Y_9$ , then  $Y_9$  back to his Shack at the intersection of  $X_9$  and  $Y_9$ .)

- (a) If Count Urr decides to represent a map of Kompleck City as a graph, how many nodes and how many edges should that graph have? (Hint, were Kompleck City

1 block on a side, 2 by 2, or 3 by 3, the number of edges and nodes would be 5 and 4, 13 and 9, and 25 and 16, respectively.)

**Answer:** There'd simply be  $10^2 = 100$  nodes (one for each intersection). For the edges, we note that each of the 20 Aves. is split into 9 sections. So there are  $20 \cdot 9 = 180$  edges that correspond to blocks of avenues, plus a single edge for The Zim Bob Way, which gives us **181 edges and 100 nodes**.

- (b) How long must Count Urr's reign last if he is to take every possible path through his realm? (Hint, you might want to try this on a smaller grid first. For example, the number of paths, were Kompleck City 1 by 1, 2 by 2, and 3 by 3, would be 2, 6, and 20, respectively.)

**Answer:** First, we note that if we were to remove The Zim Bob Way, the graph would be acyclic; it'd be impossible to ever go west or south. Thus, the number of cycles that Count Urr can take is the same as the number of paths from The Library to his Shack. The number of paths to The Library is the number of paths to the intersection of  $X_9$  and  $Y_8$  (henceforth  $(X_9, Y_8)$ ) plus the number of paths to  $(X_8, Y_9)$ . And in general, the number of paths to  $(X_n, Y_m)$  (henceforth  $P(X_n, Y_m)$ ) is  $P(X_{n-1}, Y_m) + P(X_n, Y_{m-1})$ . Thus, we have the following recurrence relation:

$$(1) \quad P(X_0, Y_m) = 1$$

$$(2) \quad P(X_n, Y_0) = 1$$

$$(3) \quad P(X_n, Y_m) = P(X_{n-1}, Y_m) + P(X_n, Y_{m-1})$$

When we compute this recurrence relation, we'll notice that we're doing the same process as when constructing Pascal's Triangle. So the number of ways to get to  $(X_n, Y_n)$  is  $\binom{2n}{n}$ . So our answer is that Count Urr's reign must last  $\binom{18}{9} = 48,620$  days or over 133 years.

Note also that in all those years, Vlad will visit the lonely points of  $(X_0, Y_9)$  and  $(X_9, Y_0)$  only once.

(2) A *undirected hyperedge* is like an ordinary undirected edge except it may be between<sup>1</sup> any number of nodes (well, any number  $\geq 2$ ). *Undirected hypergraphs* are a superset of regular undirected graphs that may contain hyper-edges (so all undirected graphs are hypergraphs). For example, I can have a graph that has 4 nodes:  $A$ ,  $B$ ,  $C$ , and  $D$ , and the following hyperedges:

- $\{A, B\}$  (This is just a regular undirected edge between  $A$  and  $B$ )
- $\{C, D\}$  (Another regular edge between  $C$  and  $D$ )
- $\{A, C, D\}$  (A hyperedge among (“between”?)  $A$ ,  $C$ , and  $D$ . Note that, since this is undirected, this is the same as the hyperedge  $\{C, D, A\}$ .)
- $\{A, B, C, D\}$  (A hyperedge between  $A$ ,  $B$ ,  $C$ , and  $D$ .)

(a) How many plain<sup>2</sup> undirected hypergraphs are there that have:

(i) 2 nodes: ( $A$  and  $B$ )

**Answer:** Only 2 graphs: Namely

**G1:**  $\{\}$  (The empty graph.)

**G2:**  $\{\{A, B\}\}$  (“Fully” connected.)

(ii) 3 nodes: ( $A$ ,  $B$ , and  $C$ )

**Answer:** 16 graphs: Namely

**G1:**  $\{\}$  (The empty graph.)

**G2:**  $\{\{A, B\}\}$

**G3:**  $\{\{A, C\}\}$

**G4:**  $\{\{B, C\}\}$

**G5:**  $\{\{A, B\}, \{A, C\}\}$

**G6:**  $\{\{A, B\}, \{B, C\}\}$

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<sup>1</sup>Grammatically and etymologically, I should write “among”, since between is specific to 2 things.

<sup>2</sup>By “plain” I mean without self loops or multiple edges. The hyperedges  $\{A, B, C\}$  and  $\{A, B\}$  wouldn’t count as multiple edges.

**G7:**  $\{\{A, C\}, \{B, C\}\}$

**G8:**  $\{\{A, B\}, \{A, C\}, \{B, C\}\}$

**G9:**  $\{\{A, B, C\}\}$

**G10:**  $\{\{A, B\}, \{A, B, C\}\}$

**G11:**  $\{\{A, C\}, \{A, B, C\}\}$

**G12:**  $\{\{B, C\}, \{A, B, C\}\}$

**G13:**  $\{\{A, B\}, \{A, C\}, \{A, B, C\}\}$

**G14:**  $\{\{A, B\}, \{B, C\}, \{A, B, C\}\}$

**G15:**  $\{\{A, C\}, \{B, C\}, \{A, B, C\}\}$

**G16:**  $\{\{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}$  (“Hyper-fully” connected.)

(iii) 4 nodes:  $(A, B, C,$  and  $D)$

**Answer:** There are  $2^{11} = 2,048$  such graphs. (To get this, I applied my answer from part b.)

(b) Write a general expression for the number of plain undirected hypergraphs that have  $n$  nodes. (Hint, you may use the  $\sum$  operator, but not ellipses “ $\dots$ ”.)

**Answer:** One way to do this would be to note that there are  $\binom{n}{i}$  possible hyperedges of degree  $i$  (each of which is either there or not), so this gives the expression as

$$2^{\binom{n}{2}} 2^{\binom{n}{3}} 2^{\binom{n}{4}} \dots 2^{\binom{n}{n}}$$

but this uses ellipses. To get rid of the ellipses, we note that this is equal to

$$2^{\binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \dots + \binom{n}{n}}$$

for which we can use the  $\sum$  operator:

$$2^{\sum_{i=2}^n \binom{n}{i}}$$

(c) BONUS: Write a *closed form* general expression for the number of plain undirected hypergraphs that have  $n$  nodes. (You can use this answer here for part b too.)

**Answer:** Another way to do this is to consider the power set of the nodes. Remember that the power set here would be the set of all subsets of nodes. Almost every element of the power set can then be considered to be a hyperedge (or not) in our hypergraph. The exceptions are those sets that have fewer than 2 elements.

There are  $2^n$  elements in the power sets, but  $n$  of these contain only 1 element, and 1 is the empty set. So the number of subsets that can be considered for being a hyperedge is  $2^n - n - 1$ , which gives the number of possible hypergraphs as

$$2^{2^n - n - 1}$$